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Fractal analysis of experimental data

Analiza fraktalna danych eksperymentalnych

Abstract

The publication discusses selected methods of analysis of experimental data used to identify deterministic chaos in the time series of quotations of selected companies listed on the Polish stock exchange. The following issues are presented: attractor reconstruction, Fourier analysis, analysis of the scaled R/S range, fractal and correlation dimension, Lyapunov exponents. The aim of the publication is to indicate selected methods for the analysis of fractal time series experimental data.

Keywords
deterministic chaos, fractal analysis, experimental data

Streszczenie

W publikacji zostały omówione wybrane metody analizy danych eksperymentalnych służących do identyfikacji chaosu deterministycznego w szeregach czasowych notowań wybranych spółek notowanych na polskiej giełdzie. Zaprezentowano następujące zagadnienia: rekonstrukcja atraktora, analiza Fouriera, analiza przeskalowanego zakresu R/S, wymiar fraktalny i korelacyjny, wykładniki Lapunowa. Celem publikacji jest wskazanie wybranych metod służących do analizy fraktalnych szeregów czasowych danych eksperymentalnych.

Słowa kluczowe
chaos deterministyczny, analiza fraktalna, dane eksperymentalne

JEL: G, G1, G17

Introduction

The complexity of phenomena on capital markets makes it impossible to detect all the variables affecting the behaviour of the system. The strange attractors described by the model equations rarely occur in reality, but they can nevertheless determine whether the system is linear or chaotic. Due to the fact that the detection of all variables describing the system is unrealistic, it becomes advisable to build a phase space on the basis of a smaller number of variables and data from past time periods.

According to Buła, an important problem related to the study of financial time series in fractal terms is the fact that their graphic representations should be included in the class of natural stochastic fractals. Due to the impossibility of a priori determination of the laws governing the fluctuations of the studied quantities, it is also impossible to calculate the fractal dimension and it is necessary to use appropriate estimation methods. So far, many different methods have been proposed, among which the author in this article, for the reasons described above, used the segmentation-variation method, the field division method and the Higuchi method. Each of the methods used represents a different philosophy of calculation: in the case of curves located on a plane, in the segment-variation method, the relationship between the area of rectangles covering a given structure in successive segments and the length of their base is analysed, in the division of the area – between successively obtained areas, and in the Higuchi method between the length of the curve and the length of the base of the segment (Buła, 2017, p. 14).

As Rzeszótko writes, the use of fractal analysis in the study of the properties of time series of stock exchange quotations allows to estimate the probability of specific movements of the capital market and to determine the dimension of the boundary sets of the dynamical system describing its behaviour. Phenomena on the financial market are so complex that it is impossible to determine all the variables of the system, but it is possible to perform the so-called phase space reconstruction (attractor reconstruction) based on the historical data of a smaller number of variables (Rzeszótko, 2016, p. 132).

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Attractor reconstruction

The construction of phase space is simple in the case of the existence of equations describing given phenomena and the knowledge of all variables related to the system. If you do not know the system variables based on the effects, you can perform a phase space reconstruction.

David Ruelle developed a method for reconstructing phase space based on a single dynamical variable. The remaining dimensions are replaced with the delayed equivalents of the observed variable. Ruelle carried out a mathematical proof and showed that the reconstructed phase space is characterized by an identical (with the true phase space of two variables) fractal dimension and the distribution of Lyapunov's exponents. From this, it can be concluded that the phase space can be reconstructed without knowing the equations of motion from experimental data. Experimental data are necessary to determine the information needed for reconstruction, e.g. the correct time delay. The information about the number of dimensions of the attractor is replaced by placing it in a space with a number of dimensions higher than the attractor itself (e.g. a plane placed arbitrarily in a three-dimensional space still has two dimensions). An attractor as a nonlinear dynamical system retains its dimension when increasing the capacitive dimension over the fractal dimension, because the points are correlated and condense at a specific place in space, regardless of the dimension. Therefore, if an attractor with a dimension higher than the actual dimension of the attractor is reconstructed, choosing the appropriate dimension is not a problem.

Estimation of the corresponding time delay can be made from an equation (Peters, 1997, p. 152):

$$m \times \tau = Q$$

where:
m – denotes the capacitive dimension,
τ – denotes the time shift,
Q – stands for the mean orbital period.

The time lag is a percentage of the orbit for each dimension. Therefore, the orbital period does not change in the higher dimensions for this quotient. The mean orbital period can be obtained on the basis of the R/S analysis by determining the time moment at which the correlation between the time courses disappears.

The reconstruction of the attractor therefore consists of experimentally trying different rules to find the most appropriate reality and gives us the ability to calculate fractal dimensions and measure sensitivity to changes in initial conditions.

Fourier analysis

Changes in the state of a dynamical system in time are described by time functions $F(t)$ in the case of continuous time or time series in the case of sampling at regular time intervals. Any time-dependent function $F(t)$ can be replaced by a superposition of its periodic components. The distribution of the function $F(t)$ into components is called the spectral distribution.

Depending on the nature of the function $F(t)$, it can be represented in two ways. A periodic function can be replaced by an assembly, a set of functions with frequencies that are integer multiples of the fundamental frequency. The linear composition of these functions is called the Fourier series.

If F(t) is a periodic function with a period T, i.e. F(t) = F(t + nT) then the frequencies of all components are integer multiples of the fundamental frequency $1/T = \omega_0/2\pi$. The Fourier series in the complex form takes the form (Baker & Gollub, 1998, p. 37):

$$F(t) = \sum_{n \to -\infty}^{\infty} (a_n e^{in\omega_0 t})$$

where:
a_n – denotes the amplitude of the n-th component with a frequency of $n\omega_0$.

The amplitude value can be obtained from the formula (Baker & Gollub, 1998, p. 38):

$$a_n = \frac{\omega_0}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} F(t) e^{-in\omega_0 t}.$$

However, non-periodic functions are more common, for which the spectrum is expressed by oscillations from the frequency continuum. Such a spectrum is called the Fourier transform. This type of representation of a time-varying function is of particular importance in the case of

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dynamical chaotic systems. The Fourier transform is the general form of the Fourier series. Since it is assumed that the period T of the function $F(t)$ can be infinitely large, the function $F(t)$ can be described even if it is not periodic. In the case of non-periodic functions, the discrete set of component frequencies becomes a continuous set of spectral frequencies. The amplitude of the component an is replaced by the Fourier transform $a(\omega)\delta\omega$.

A Fourier series transforms into a Fourier transform as a result of the transformation:

$T \to \propto$,
$n\omega_0 \to \omega$,
$a_n \to a(\omega)d\omega$.

Fourier transform $a(\omega)$ is often in a complex form, then a real function is defined, the so-called power spectrum (Baker & Gollub, 1998, p. 40):

$S(\omega) = |a(\omega)|^2$.

With the help of the Fourier transform, we can analyse time series, the properties of which we do not know. If the time series is complicated, then the determination of the Fourier transform is also complicated.

Analysis of the scaled R/S range

Alternative method to determine the time delay τ consists in determining the Hurst exponent. If we study a system with many degrees of freedom, we usually assume that the changes occurring in this system are random walking (the nature of Brownian motion). By studying changes in the state of water in artificial water bodies, Hurst tested this assumption and found that fluctuations in water levels do not create a time series in which the range of fluctuations is proportional to $t^{1/2}$ (this is the case with Brownian motion). Dividing the range of fluc-tuations by the standard deviation of the obser-vation, he created a dimensionless H-index (*rescaled range analysis*), abbreviated as R/S analysis (Peters, 1997, p. 64).

The Hurst exponent is determined as follows. The measurement data (in the form of sequential samples) are divided into intervals with a fixed number of points equal to N, as shown in Figure 1.

For each of the intervals shown in Figure 1, we define a number of characters (Peters, 1997, p. 65):

$$T_i = \sum_{j=1}^{i}(x_j - \overline{x}_N) = \sum_{j=1}^{i} x_j - i \cdot \overline{x}_N$$

where:
T_i – denotes the cumulative deviation in N samples,
x_j – sample value at time j,
$\bar{x}_N$ – the arithmetic mean of the data for N samples.

For each of the intervals of the above series, we calculate the quantity R called the range:

$R = \max(T_i) - (\min(T_i)$

and the standard deviation S.

The R/S characteristic for the entire time series is determined as the arithmetic mean of R/S determined for all intervals of length N.

For the next values of N, the graph shown in Figure 2 is built. The directional coefficient of the tangent to the graph ln(R/S) as a function of ln(N) determines the value of the coefficient H. In practice, the tangent coefficient is replaced by the directional coefficient of simple regression determined for a given interval of changes in the number of intervals N. If the number N includes too many measurement points, then the process resembles random wandering (long-term memory disappears – memory between successive intervals). This point corresponds to the end of the natural period of the physical system (Figure 2 – log(N^*)). At point N^* the curve ln(R/S) changes the slope as shown in Figure 2, for stochastic signals $H = 0.5$.

Magnitude N^* allows you to specify the size of the time shift τ necessary for the reconstruction of the attractor. Size τ is determined from the following relationships:

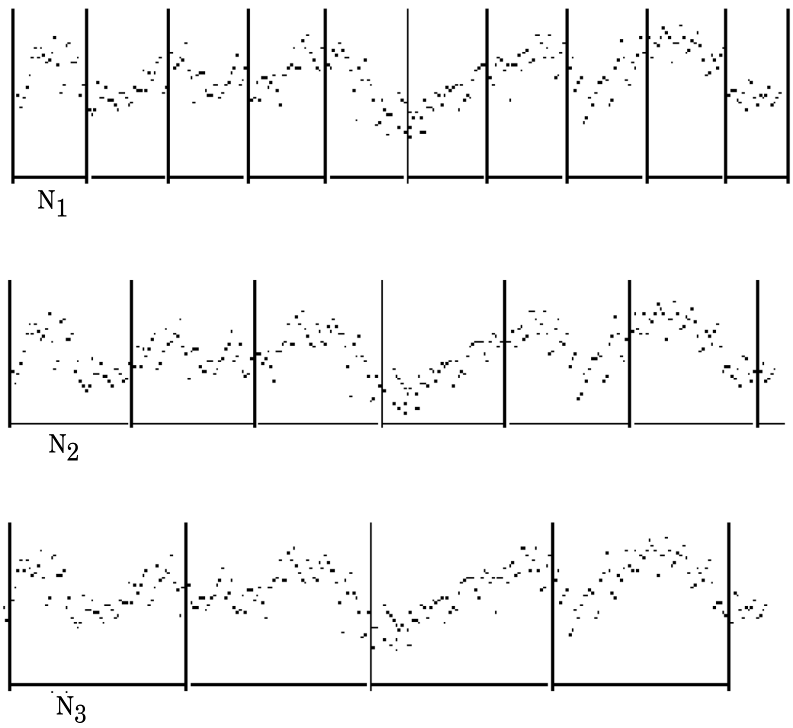
$$\tau = \frac{N^*}{n}$$

where:
n – immersion dimension of the reconstructed attractor.

From the point of view of Zieliński, "The minimum immersion dimension is the minimum number of geometric coordinates needed to faithfully represent the entire attractor. It is also the number of degrees of freedom (the number of independent variables) needed to represent a dynamic system" (Zieliński, 2000, p. 255).

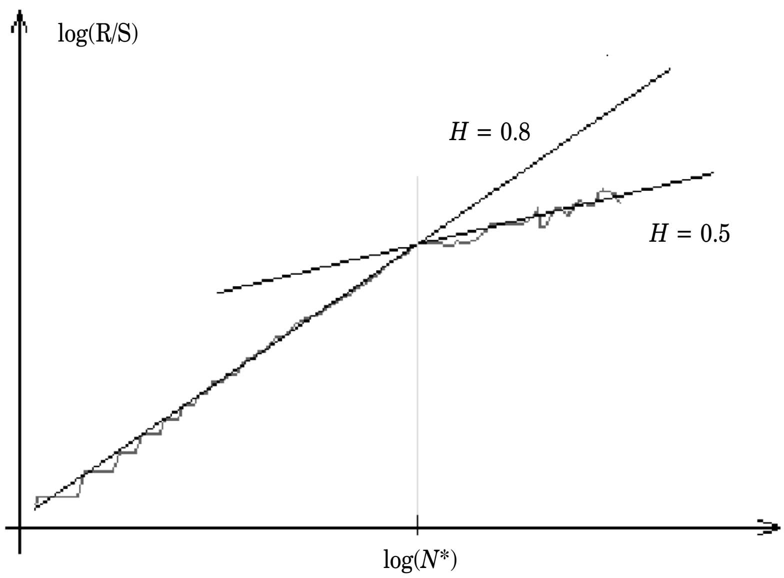
The Hurst exponent can be used in analyses of various time series with a small number of assumptions about the studied system. On its basis, it becomes possible to distinguish random series from non-random series. There are three classes of magnitude of the Hurst exponent (Peters, 1997, p. 67):

Figure 1. Hurst exponent principle



Source: Siemieniuk, 2001, p. 144.

Figure 2. Hurst's exponent



Source: Siemieniuk, 2001, p. 145.

- $H = 0.5$ at which the time series is random,
- $0 \le H < 0.5$ time series is antipersistent or ergodic ("relapsing to mean"),
- The $0.5 < H < 1$ time series is persistent, i.e. strengthening the trend.

Capital markets are persistent series and are fractals. The events are correlated on all time scales, which means that the probabilities of two consecutive events with the same sign are not equal, and the fractal dimension of the probability

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distribution is expressed by a number between 1 and 2. In time series, for which the Hurst exponent is approaching unity, more and more consecutive observations have the same sign. As H increases, the line becomes smoother and smoother, there is less noise in the system, trends become visible, deviations from the mean increase. The Hurst exponent is a measure of the complexity of a fractal series.

Applying R/S analysis to different capital markets can show that they have a fractal structure, and the presence of non-periodic cycles indicates that they are non-linear phenomena.

Fractal and correlational dimension

The trajectories of a chaotic system in phase space do not form any single geometric object such as a circle or a torus, but instead form objects called strange attractors, whose structure resembles that of a fractal. The fractal dimension describes how a system fills its space and is the result of all the factors affecting this system.

Most often, an object is placed in a space whose dimension is greater than the fractal dimension of the object. Such a space is called the capacitive or topological dimension. Placing a fractal object in a capacitive dimension greater than its fractal dimension does not change its dimension. The fractal dimension depends on how an object or a time series fills its space. To determine the fractal dimension for a given structure, it is necessary to measure how it condenses in space.

For experimental data, the correlation dimension D2 is determined, which defines the expression of the form (Schuster, 1993, p. 133):

$$D_2 = \lim_{l \to 0} \frac{1}{\ln l} \ln \sum_i p_i^2$$

$$\sum_i p_i^2 \approx \lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j} \Theta\left(1 - |x_i - x_j|\right) = C_2(1)$$

where:

Θ – denotes the Heaviside step function determining the number of pairs of attractor points whose distance is less than l.

The quantities K_2 and D_2 are related by a relation (Schuster, 1993, p. 133):

$$\lim_{l \to 0} \lim_{n \to \infty} \ln C_2^n\,(l) = D_2 \ln l + nK_2$$

where:

n – immersion dimension.

The fractal dimension is important information about the system because it allows you to determine the minimum number of dynamic variables needed to describe the system. At the same time, the fractal dimension is the lower limit of the number of possible degrees of freedom.

The American, German, and British markets have a fractal dimension between 2 and 3 (Peters, 1997, p. 165). It follows that three variables are sufficient to describe these markets. In the case of more complex markets, e.g. the Japanese market (fractal dimension 3.05), a larger number of variables is needed to describe its behaviour.

Lyapunov exponents

Chaotic dynamic systems are distinguished by high sensitivity to initial conditions. As a result of this property, initially close trajectories diverge exponentially in a finite time interval. An important way to determine the sensitivity of systems to initial conditions is the Lyapunov analysis, which allows the detection of the presence of this property in experimental signals and dynamical systems.

For measurement data in the form of a time series:

$$\{x_n\} = \{x_1, x_2, x_3, x_4, \ldots, x_n\}$$

it is impossible to determine all the coefficients of Lyapunov. However, it is possible to determine the value of the largest Lyapunov exponent. In this case, on an attractor immersed in D dimensional space, two points separated by at least one orbital period are selected. The distance of these points is $L(\tau_j)$. Then the distance of the selected points is calculated after a certain time of evolution. The new distance of the pair of points is $L'(\tau_{j+1})$. The largest Lyapunov exponent is calculated according to the formula (Peters, 1997, p. 156):

$$L_1 = \frac{1}{\tau} \sum_{j=1}^{m} \log_2 \frac{L'(\tau_{j+1})}{L(\tau_j)}$$

where:

m – denotes the capacitive dimension,

τ – denotes the time delay.

Determination of the largest Lyapunov exponent is possible when such characteristics of the attractor as fractal dimension, mean orbital time and time delay are known. For long time series, the results of L_1 value calculations tend to a stable value, which is an estimate of the magnitude of the largest Lyapunov exponent.

Determining the largest exponent of Lyapunov is possible if we know the fractal dimension of the attractor. On the other hand, in order to determine the fractal dimension of the attractor, it is necessary to know the size of the time delay – the value necessary to reconstruct the attractor.

The Lyapunov exponent indicates whether the behaviour of a dynamical system is chaotic. It can also be defined in the form (Schuster, 1993, p.35):

$$\lambda(z_o) = \lim_{N\to\infty}\lim_{\varepsilon\to 0}\frac{1}{N}\ln\left|\frac{f^N(z_o+\tau)-f^N(z_o)}{\tau}\right|.$$

Parameter f^N in the above formula is the N fold composition of the function f occurring in the equation describing the behaviour of a dynamical system. It is a measure of the change in distance between two initial points (z_o) and ($z_o+\tau$) hat evolve through N iterations. The Lyapunov coefficient is also a measure of the loss of information about a system in one iteration (Schuster, 1993, p. 35). Where the $\lambda > 0$, a dynamical system is said to have chaotic behaviour.

If the same starting point is determined twice for a dynamical system, then the Lyapunov exponent will determine the average increase in errors between the values of the point for a given time T. The exponent is calculated for each dimension, if any of the exponents is equal to 0, then ordinary differential equations are sufficient to describe the system. If any of the exponents of the Lyapunov system is greater than zero, it is assumed that the system is chaotic.

Conclusions

Fractal analysis of experimental data is an important complement to classical methods of analysis. It should be remembered that fractal analysis does not bring us closer to forecasting the value of a parameter for a specific day, but it allows us to assess the probability of specific system behaviours and model alternative scenarios of its behaviour. In order to subject the collected data to fractal analysis using methods based on deterministic chaos theory, which are used, among other things, to eliminate trend lines (attractor reconstruction, Fourier analysis) and estimation of deterministic chaos measures (Hurst exponents, fractal dimension and correlation dimension, Lyapunov exponents).

One of the basic methods to distinguish random series from non-random series is the analysis of the scaled R/S range, which is a method of determining the time delay and consists in determining the Hurst exponent. It is used to determine the effects of long-term memory and detect fractional Brownian motion.

The fractal dimension describes how a system fills its space and is the result of all the factors affecting this system. Most often, an object is placed in a space whose dimension is greater than the fractal dimension of the object. Such a space is called the capacitive or topological dimension. Placing a fractal object in a capacitive dimension that is larger than its fractal dimension does not change its dimension. The fractal dimension depends on how an object or time series fills its space. The fractal dimension is important information about a system because it allows you to estimate the correlation dimension, which is the minimum number of dynamic variables needed to describe a given system. The smaller the number, the less complicated and more predictable the system is.

Lyapunov's exponents are one of the tools of deterministic chaos theory that measure the sensitivity of dynamical systems to changes in initial conditions. A positive exponent measures the stretching of a state space, that is, it determines how a prediction based on an inaccurate estimate of the output will deviate from the actual evolution of the system. In the second approach, the Lyapunov exponent allows us to determine the time after which the value of our forecasts will disappear.

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